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# The Behavior of Quantization Spectra as a Function of Signal-to-Noise Ratio

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*An expression for the spectrum of quantization error in a discrete-time system whose input is a sinusoid plus white Gaussian noise is derived. This quantization spectrum consists of two components: a white-noise floor and spurious harmonics. The dithering effect of the input Gaussian noise on both components of the spectrum is considered. Quantitative results in a discrete Fourier transform (DFT) example show the behavior of spurious harmonics as a function of the signal-to-noise ratio (SNR). These results have strong implications for digital reception and signal analysis systems. At low SNRs, spurious harmonics decay exponentially on a log-log scale, and the resulting spectrum is white. As the SNR increases, the spurious harmonics figure prominently in the output spectrum. A useful expression is given that roughly bounds the magnitude of a spurious harmonic as a function of the SNR.*

## I. Introduction

This work was inspired by consideration of a 2-million channel spectrum analyzer built by the Digital Projects Group of the Communications Systems Research Section [1]. This spectrum analyzer is a prototype of a larger system that will be used in the sky-survey portion of the Search for Extraterrestrial Intelligence (SETI) project. After computer simulations were performed, 8-bit input quantization was observed to pose the greatest limitation to the dynamic range of the spectrum analyzer. This is

because quantization is a nonlinear process that generates spurious harmonics in the spectrum of the quantizer output.

Previous work by Bennett [2] considered the spectra of quantized signals when the system input has "energy uniformly distributed throughout a definite frequency band and with the phases of the components randomly distributed." Hurd [3] developed an expression for the correlation function of a quantized sine wave plus Gaussian noise and examined the case where the input noise spectrum is rectangular narrow-band and the signal-to-noise ratio (SNR) is small. Quantization error spectra are most commonly assumed to be white [4]. This article derives an

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expression for the spectrum of a quantized sine wave plus white Gaussian noise. An SNR transition region where the spectrum goes from being filled with spurious harmonics to white is presented. This transition is due to the dithering effect of the input Gaussian noise. A rule of thumb is given bounding the size of a spurious harmonic as a function of the SNR. Implications for digital reception and signal analysis systems are considered.

## II. Power Spectrum of the Quantization Error

Consider the quantizer system in Fig. 1 with input  $x$  and output  $y$ . One can write:

$$y = Q[x] = x - e \quad (1)$$

where  $Q[\ ]$  is the quantization operator and  $e$  is the quantization error. When  $Q[\ ]$  is a uniform mid-tread symmetric quantizer with a staircase input-output relation as in Fig. 2,  $e$  can be expressed as a sawtooth function of  $x$  as in Fig. 3. Assuming an infinite quantizer (or equivalently, no quantizer saturation), one can write a Fourier series expansion for  $e(x)$  as in [5]:

$$e(x) = -\frac{\Delta}{j2\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k}{k} \exp\left(\frac{j2\pi kx}{\Delta}\right) \quad (2)$$

Now consider the system in Fig. 4 where the input is  $A \sin(\omega_0 t + \phi) + z(t)$  and  $z(t)$  is zero mean, Gaussian noise with variance  $\sigma^2$ . The continuous-time signal  $e(t)$  can be written as

$$e(t) =$$

$$-\frac{\Delta}{j2\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k}{k} \exp\left(\frac{j2\pi kx}{\Delta} (A \sin(\omega_0 t + \phi) + z(t))\right) \quad (3)$$

The autocorrelation function of  $e(t)$  is defined as

$$R_e(t, t + \tau) = E\{e(t)e(t + \tau)\} \quad (4)$$

Returning to Fig. 4, one sees that  $x[n] = e(nT)$  is a discrete-time random process. The autocorrelation function of  $x[n]$  can be expressed as

$$\begin{aligned} R_x[n, n + k] &= E\{x[n]x[n + k]\} \\ &= E\{e(nT)e(nT + kT)\} = R_e(nT, nT + kT) \end{aligned} \quad (5)$$

If the phase  $\phi$  of the input sinusoid is a random variable uniformly distributed between 0 and  $2\pi$ ,  $\bar{R}_x[k]$  can be computed as

$$\bar{R}_x[k] = \frac{1}{2\pi} \int_0^{2\pi} R_x[n, n + k] d\phi \quad (6)$$

Leaving the details to Appendix A, one obtains

$$\bar{R}_x[k] = \begin{cases} \frac{\Delta^2}{12} + \frac{\Delta^2}{\pi^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^2} \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} l^2\right) J_0\left(\frac{2\pi A l}{\Delta}\right); & k = 0 \\ \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} S_n \exp(jn\omega_0 T k); & k \neq 0 \end{cases} \quad (7)$$

where

$$S_n = \frac{\Delta^2}{\pi^2} \left( \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} l^2\right) J_n\left(\frac{2\pi A l}{\Delta}\right) \right)^2 \quad (8)$$

and  $J_n(x)$  is the  $n$ th-order Bessel function. The power spectrum of the discrete-time random process  $x[n]$  is

$$S(\omega) = \sum_{k=-\infty}^{\infty} \bar{R}_x[k] e^{j\omega T k} \quad (9)$$

Finally, the power spectrum can be expressed as

$$S(\omega) = N_Q + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} 2\pi S_n \delta((\omega T - n\omega_0 T) \bmod 2\pi) \quad (10)$$

where

$$N_Q = \frac{\Delta^2}{12} + \left( \frac{\Delta^2}{\pi^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^2} \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2} l^2\right) \times J_0\left(\frac{2\pi A l}{\Delta}\right) \right) - \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} S_n$$

and  $S_n$  is defined in Eq. (8) (details of this derivation are left to Appendix A). The  $N_Q$  term in Eq. (10) represents a frequency-independent quantization noise floor (see Appendix A). The infinite sum in Eq. (10) specifies the phase-averaged magnitude and location of all spurious harmonics in the frequency domain. Depending on the value of  $\omega_0 T$ , spurious harmonics can be spread throughout the frequency domain or lie concentrated at only a few frequencies. This complicates digital reception of weak signals in the presence of stronger interferers. While a strong interferer may be easily identified and filtered out, the spurious harmonics generated by a strong interferer would require more complex filtering techniques. A spectrum (taken from a discrete Fourier transform) with spurs is shown in Fig. 5.

When  $\sigma > \Delta$ ,

$$S(\omega) = \frac{\Delta^2}{12} + O\left(\exp\left(\frac{-2\pi^2\sigma^2}{\Delta^2}\right)\right) + \langle \text{spur term} \rangle$$

where the  $\langle \text{spur term} \rangle$  is a delta function with weight zero or weight  $O(\exp(-4\pi^2\sigma^2/\Delta^2))$ , depending on whether or not a spur was located at that frequency. These results are consistent with those in [5].

With the exception of the  $\Delta^2/12$  term, all the components in the power spectrum in Eq. (10) are dependent on the ratio  $\sigma/\Delta$ . When this ratio is much greater than 1, the resulting spectrum is essentially  $\Delta^2/12$  and white. In this manner, the input Gaussian noise has a dithering effect on the spectrum. For other values of  $\sigma/\Delta$ , it is not im-

mediately obvious how the spectrum will appear. For this reason, an example involving a discrete Fourier transform (DFT) is presented in the next section.

### III. DFT Example

This example provides a quantitative analysis of the manner in which spurious harmonics are dithered due to additive white Gaussian noise. In particular, an SNR region where the magnitude of a spurious harmonic decays exponentially on a log-log scale is presented. Consider the DFT of the signal  $x[n]$  in Fig. 4. One can write

$$E\{|X[k]|^2\} = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} E\{x[n]x[r]\} e^{\frac{-j2\pi k(n-r)}{N}} \quad (11)$$

For the purpose of exposition, the  $\omega_0 T$  product is chosen to equal  $\pi/8$ , and the value of  $k$  is  $3N/16$ . Thus, a case is analyzed where the infinite number of spurious harmonics are aliased into only a few frequency bins. In this example, the phase is not treated as a random variable as was done in the previous section. Treating the phase as a constant allows a more general phase-dependent solution to be obtained.

By observing bin  $3N/16$ , one is examining the spectral sample that contains the spurious harmonic specified by an arrow in Fig. 5. The spectrum in Fig. 5 was generated using the following parameters:  $A = 0.5$ ,  $\phi = 0.8147576$ ,  $N = 1024$ ,  $\Delta = 1/127$ , and  $\text{SNR} = 10 \log_{10}(A^2/2\sigma^2) = 50$  dB.

Leaving the details to Appendix B, one can evaluate Eq. (11):

$$Y = E\{|X[3N/16]|^2\} = N_{Q1} + N_{Q2} + N_{Q3} + S_Q \quad (12)$$

where

$$N_{Q1} = \frac{\Delta^2}{12N} \quad (13)$$

and

$$\left. \begin{aligned}
N_{Q2} &= \frac{\Delta^2}{16\pi^2 N} \sum_{m=0}^7 \sum_{l=1}^{\infty} \frac{(-1)^l \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} l^2\right)}{l^2} \left[ \cos\left(\frac{2\pi A \Theta_m l}{\Delta}\right) + \cos\left(\frac{2\pi A \Xi_m l}{\Delta}\right) \right] \\
N_{Q3} &= -\frac{\Delta^2}{16\pi^2 N} \sum_{m=0}^7 \left[ \left( \sum_{l=1}^{\infty} \frac{(-1)^l \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} l^2\right)}{l} \sin\left(\frac{2\pi A \Theta_m l}{\Delta}\right) \right)^2 \right. \\
&\quad \left. + \left( \sum_{l=1}^{\infty} \frac{(-1)^l \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} l^2\right)}{l} \sin\left(\frac{2\pi A \Xi_m l}{\Delta}\right) \right)^2 \right] \\
S_Q &= \frac{\Delta^2}{\pi^2} \left| \sum_{l=1}^{\infty} \frac{(-1)^l \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} l^2\right)}{l} F\left(16\pi, 3, 16, \frac{2\pi A l}{\Delta}\right) \right|^2
\end{aligned} \right\} \quad (14)$$

with  $F(\cdot)$ ,  $\Theta_m$ , and  $\Xi_m$  as defined in Appendix C.

The quantities in Eq. (12) are easily computed. In fact, for the SNRs of interest, only a small number of terms is required to adequately represent the infinite sums over  $l$  (see Appendix D for more details). The spectral sample  $Y$  consists of two types of components: a white-noise-floor term and a spurious-harmonic term. As seen in Appendix B, the values of  $N_{Q1}$ ,  $N_{Q2}$ , and  $N_{Q3}$  are independent of bin number (frequency), and thus represent a white-noise floor. As a check, these quantities obey the  $1/N$  (where  $N$  is the number of points in the DFT) processing rule for white noise in a DFT. The value  $S_Q$  represents the spurious harmonic component of the spectral sample. This term is independent of the DFT size  $N$  and is responsible for the spectral sample becoming large when the SNR is high. In effect, this limits the dynamic range of the digital spectrum analyzer.

Figures 6(a) and (b) show how the value of the spectral sample changes as the SNR varies. The units are decibels relative to the carrier (dBc). The following parameters are held constant:  $A = 0.5$ ,  $\phi = 0.8147576$ ,  $N = 1024$ , and  $\Delta = 1/127$ . The quantizer step size  $\Delta$  was chosen to simulate an 8-bit input quantizer ( $\Delta \approx 2^{-B+1}$  where  $B = 8$ ). An experimental curve (simulation) is presented with the theoretical curve to show the excellent agreement. For each SNR used in the experiments, 10,000 spectra were accumulated and rescaled.

The horizontal line in Fig. 6(a) indicates the value of  $N_{Q1}$ . This would be the value of the spectral sample under traditional quantization error assumptions [4]. Below 40 dB SNR (for the 8-bit input quantizer), the value of the spectral sample  $Y$  reduces to essentially  $N_{Q1}$ . Above 70 dB SNR, the value of the spectral sample does not change by more than a few decibels. The transition region in Fig. 6(a) coincides with  $\sigma/\Delta$  approaching and exceeding unity.

Figure 7 plots  $S_Q$ , the spurious harmonic portion of the spectral sample, as a function of SNR. The units are decibels relative to the carrier (dBc) with the same conditions as above. When  $\sigma/\Delta$  approaches unity, this value decays exponentially on the log-log scale. From Eq. (14), one can estimate the behavior of  $S_Q$  when  $\sigma/\Delta$  exceeds unity to obtain a rule of thumb. Assume that the first term in the infinite sum over  $l$  in Eq. (14) dominates, and bound  $F(\cdot)$  by its maximum value (from Appendix C,  $|F(\cdot)| \leq 1$ ). This provides a rough bound on the maximum value of  $S_Q$ . In particular, compute  $S_{QdBc} = 10 \log_{10}(2S_Q/A^2)$ . Recall that  $\text{SNR} = 10 \log_{10}(A^2/2\sigma^2)$ . After manipulating,

$$S_{QdBc} = 10 \log_{10} \left( \frac{2\Delta^2}{\pi^2 A^2} \right) - \frac{20\pi^2}{\ln 10} \frac{A^2}{\Delta^2} 10^{-\frac{\text{SNR}}{10}} \quad (15)$$

This expression is easily inverted to express the SNR as a function of  $S_{QdBc}$ . As seen in Fig. 7, the rule of thumb

in Eq. (15) nicely describes the behavior of  $S_Q$  even when  $\sigma/\Delta$  is less than one. Asymptotically, Eq. (15) levels off at high SNRs to a constant value, which is consistent with empirical observations. This expression should prove useful to system designers concerned with dynamic-range limitations imposed by input quantization.

#### IV. Conclusions

Spurious harmonics pose complex filtering problems for digital reception systems. These harmonics also limit the

dynamic range of digital spectrum analyzers. Expressions have been obtained describing the spectrum of quantization error when the input is a noisy sinusoid. An example involving a DFT has provided quantitative information about the behavior of spurious harmonics in the frequency domain as a function of the SNR. The input Gaussian noise dithers the output spectrum when the ratio  $\sigma/\Delta$  exceeds 1, where  $\sigma^2$  is the noise variance, and  $\Delta$  is the step size in the input quantizer. A useful rule of thumb has been derived that roughly bounds the magnitude of a spurious harmonic as a function of the SNR.

#### Acknowledgment

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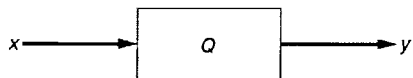


Fig. 1. Quantizer system diagram.

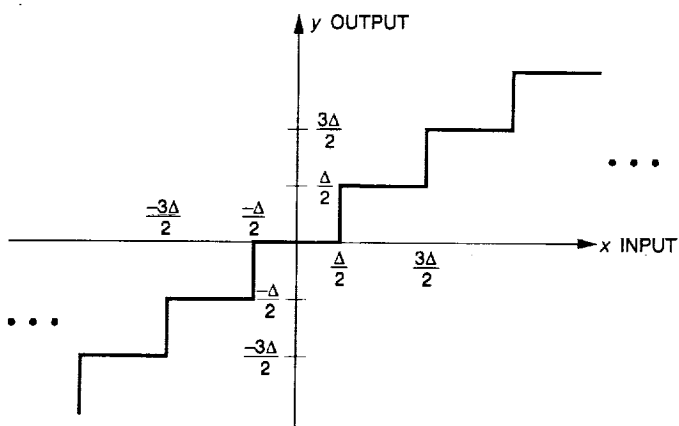


Fig. 2. Quantizer Input-output relation.

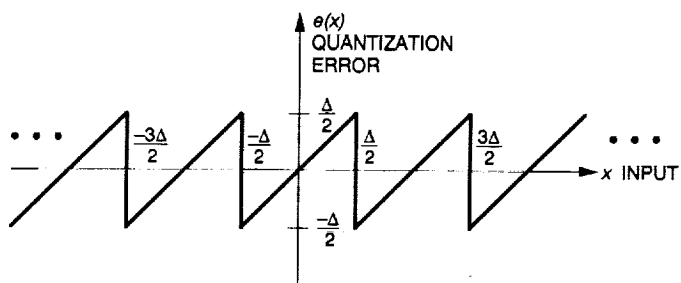


Fig. 3. Quantization error Input-output relation.

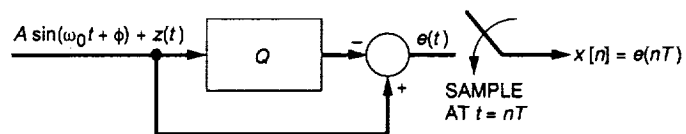


Fig. 4. Generation of discrete-time quantization error.

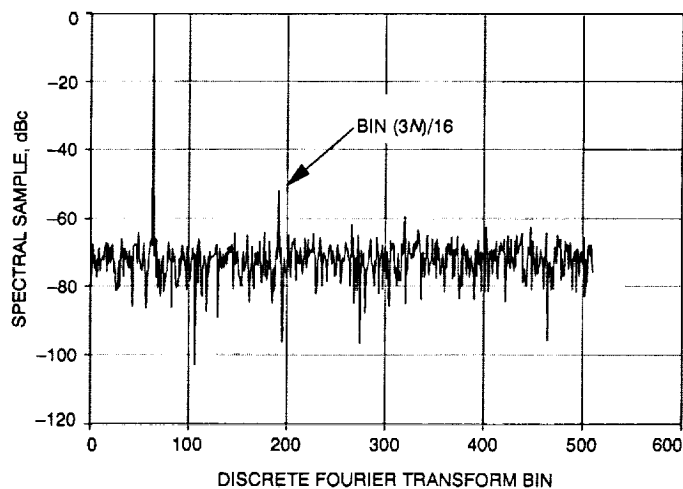
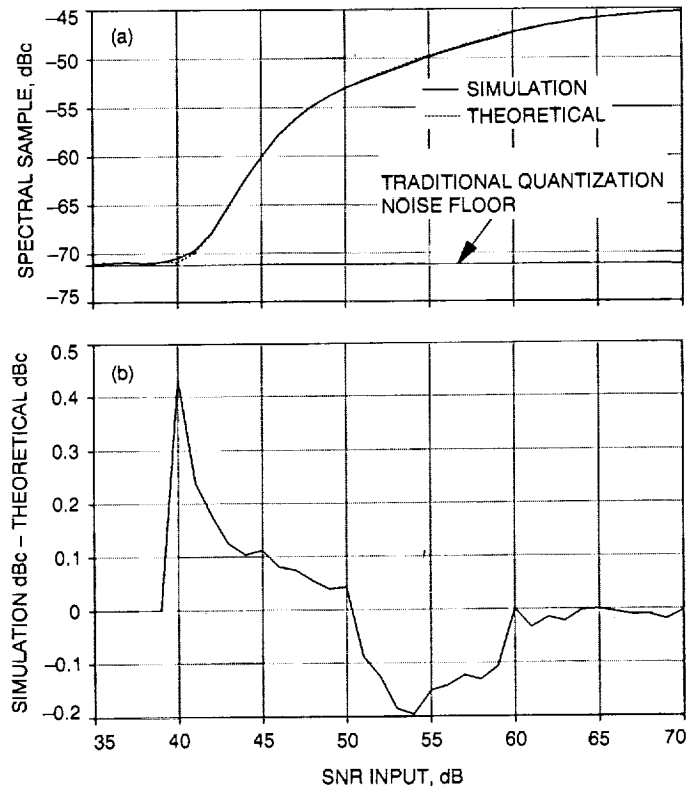
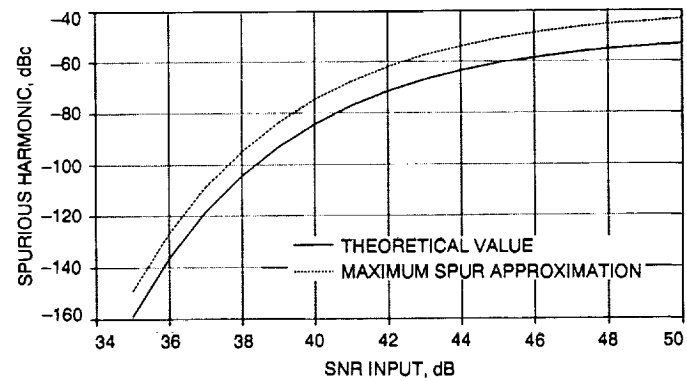


Fig. 5. DFT spectrum, 50-dB SNR Input.



**Fig. 6. Spectral sample versus (a) quantizer input SNR, and (b) quantizer input SNR, dBc difference.**



**Fig. 7. Spurious harmonics versus quantizer input SNR.**

## Appendix A

### Derivation of the Power Spectrum

From Eq. (4), first consider  $\tau = 0$ :

$$R_e(t, t) = -\frac{\Delta^2}{4\pi^2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^{l+k}}{lk} \\ \times \exp\left(\frac{j2\pi A}{\Delta}(l+k)\sin(\omega_0 t + \phi)\right) \\ \times E\left\{\exp\left(\frac{j2\pi(l+k)}{\Delta}z(t)\right)\right\}$$

Using the characteristic function of a zero-mean Gaussian random variable  $z$  with variance  $\sigma^2$ , it is known that  $E\{\exp(j\alpha z)\} = \exp(-\alpha^2/2\sigma^2)$  [6]. Therefore,

$$R_e(t, t) = -\frac{\Delta^2}{4\pi^2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^{l+k}}{lk} \\ \times \exp\left(\frac{j2\pi A}{\Delta}(l+k)\sin(\omega_0 t + \phi)\right) \\ \times \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2}(l+k)^2\right)$$

Changing indices so that  $l+k=m$  and translating the condition  $k \neq 0$  into  $m \neq l$ , one obtains

$$R_e(t, t) = \\ \frac{\Delta^2}{4\pi^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^m}{l(l-m)} \\ \times \exp\left(\frac{j2\pi Am}{\Delta}\sin(\omega_0 t + \phi)\right) \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2}m^2\right)$$

Separating the case when  $m = 0$  and rearranging the order of summation,

$$R_e(t, t) = \frac{\Delta^2}{4\pi^2} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{1}{l^2} + \frac{\Delta^2}{4\pi^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} (-1)^m \\ \times \exp\left(\frac{j2\pi Am}{\Delta}\sin(\omega_0 t + \phi)\right) \sum_{\substack{l=-\infty \\ l \neq 0 \\ l \neq m}}^{\infty} \frac{1}{l(l-m)}$$

Noting that the sum involving  $1/l^2$  evaluates to  $\pi^2/3$  [7] and (after partial fraction manipulation) that the sum involving  $1/l(l-m)$  evaluates to  $2/m^2$ , one can write

$$R_e(t, t) = \frac{\Delta^2}{12} \\ + \frac{\Delta^2}{2\pi^2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{(-1)^m \exp\left(\frac{j2\pi Am}{\Delta}\sin(\omega_0 t + \phi)\right)}{m^2} \\ \times \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2}m^2\right) \quad (\text{A-1})$$

Now consider  $\tau \neq 0$ :

$$R_e(t, t + \tau) = \\ -\frac{\Delta^2}{4\pi^2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^{l+k}}{lk} \\ \times \exp\left(\frac{j2\pi A}{\Delta}[k\sin(\omega_0 t + \phi) + l\sin(\omega_0 \tau + \phi)]\right) \\ \times E\left\{\exp\left(\frac{j2\pi}{\Delta}(kz(t) + lz(t + \tau))\right)\right\}$$

Since the input noise is white,  $z(t)$  and  $z(t + \tau)$  are independent random variables ( $\tau \neq 0$ ), and the expectation of the product becomes the product of the expectations. When one uses the same characteristic function method detailed above,

$$\begin{aligned}
R_e(t, t + \tau) = & -\frac{\Delta^2}{4\pi^2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^{l+k}}{lk} \\
& \times \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2}(l^2 + m^2)\right) \\
& \times \exp\left(\frac{j2\pi A}{\Delta}[k \sin(\omega_0 t + \phi) \right. \\
& \left. + l \sin(\omega_0 t + \omega_0 \tau + \phi)]\right) \quad (\text{A-2})
\end{aligned}$$

As indicated in Eq. (5), to obtain the discrete-time autocorrelation function  $R_x[n, n+k]$ , replace  $t$  with  $nT$  and  $\tau$  with  $kT$  in Eqs. (A-1) and (A-2). The evaluation of Eq. (6) involves changing the order of integration and summation until only terms involving the phase  $\phi$  are inside the integral. It is useful at this time to use the Jacobi-Anger formula [7]:

$$e^{jx \sin \phi} = \sum_{p=-\infty}^{\infty} J_p(x) e^{jp\phi}$$

One can now evaluate

$$\begin{aligned}
\bar{R}_x[0] = & \frac{\Delta^2}{12} + \frac{\Delta^2}{2\pi^2} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^l}{l^2} \\
& \times \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2}l^2\right) \sum_{p=-\infty}^{\infty} J_p\left(\frac{2\pi Al}{\Delta}\right) e^{jp\omega_0 T n} \\
& \times \frac{1}{2\pi} \int_0^{2\pi} e^{jp\phi} d\phi
\end{aligned}$$

The integral above will be 1 when  $p = 0$  and 0 otherwise. Noting that the resulting expression inside the above sum is an even function of  $l$ , one obtains the first half of Eq. (7). In evaluating  $\bar{R}_x[k]$  when  $k \neq 0$ , one again employs the Jacobi-Anger formula and interchanges the order of summation and integration so that the appropriate terms from Eq. (A-2) are inside the integral. Using the orthogonality of exponentials and noting that  $J_p(x) = (-1)^p J_{-p}(x)$ , one obtains

$$\begin{aligned}
\bar{R}_x[k] = & - \sum_{p=-\infty}^{\infty} (-1)^p \exp(-jp\omega_0 T k) \frac{\Delta^2}{4\pi^2} \\
& \times \left( \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \frac{(-1)^l J_p\left(\frac{2\pi Al}{\Delta}\right)}{l} \exp\left(-\frac{2\pi^2\sigma^2}{\Delta^2}l^2\right) \right)^2
\end{aligned}$$

Consider the sum over  $l$ . When  $p$  is even, this sum will be zero since  $J_p(x) = (-1)^p J_p(-x)$ . When  $p$  is odd, the sum over  $l$  is an even function of  $l$ , and one can reduce the double-sided infinite sum to a single-sided infinite sum. By changing the index from  $p$  to  $-n$ , one obtains the second half of Eq. (7).

In evaluating Eq. (9), one can write

$$S(\omega) = \bar{R}_x[0] + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \bar{R}_x[k] e^{j\omega T k}$$

Note that the first term,  $\bar{R}_x[0]$ , is independent of the frequency,  $\omega$ . Using the notation in Eq. (7),

$$\begin{aligned}
S(\omega) = & \bar{R}_x[0] \\
& + \left( \sum_{k=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} S_n \exp(jn\omega_0 T k) \exp(j\omega T k) \right) \\
& - \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} S_n
\end{aligned}$$

Note that the third term above is independent of frequency. Writing out the expression for  $\bar{R}_x[0]$ , one arrives at Eq. (10) after noting

$$\sum_{k=-\infty}^{\infty} e^{jxk} = 2\pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k) = 2\pi \delta(x \bmod 2\pi) \quad (\text{A-3})$$

where  $\delta(\cdot)$  is the Dirac delta function.

## Appendix B

### Derivation of the DFT Problem

From Eq. (11), one can write

$$Y = E\{|X[k]|^2\}$$

$$\begin{aligned} &= \frac{1}{N^2} \sum_{n=0}^{N-1} E\{x[n]x[n]\} \\ &+ \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{\substack{r=0 \\ r \neq n}}^{N-1} E\{x[n]x[r]\} \exp\left(\frac{-j2\pi k(n-r)}{N}\right) \end{aligned} \quad (\text{B-1})$$

Consider the first term above. Note that it is independent of the bin number  $k$  (i.e., independent of frequency). Recalling the results of Eqs. (5) and (A-1), rearranging the order of summation, and using the Jacobi-Anger formula leaves one with

$$\begin{aligned} &\frac{1}{N^2} \sum_{n=0}^{N-1} R_e(nT, nT) = \frac{\Delta^2}{12N} \\ &+ \frac{\Delta^2}{2\pi^2 N} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{(-1)^m}{m^2} \\ &\times \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} m^2\right) \sum_{p=-\infty}^{\infty} J_p\left(\frac{2\pi A m}{\Delta}\right) e^{jp\phi} \\ &\times \frac{1}{N^2} \sum_{n=0}^{N-1} e^{\frac{jp\pi n}{8}} \end{aligned}$$

Recall that  $\omega_0 T = \pi/8$  and  $k = 3N/16$  in this example. The sum over  $n$  is nonzero only when  $p = 0 \bmod 16$ . So,

$$\frac{1}{N^2} \sum_{n=0}^{N-1} R_e(nT, nT) = N_{Q_1} + N_{Q_2}$$

where  $N_{Q_1} = \Delta^2/12N$ , and

$$\begin{aligned} N_{Q_2} &= \frac{\Delta^2}{2\pi^2 N} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{(-1)^m}{m^2} \\ &\times \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} m^2\right) \sum_{p=-\infty}^{\infty} J_{16p}\left(\frac{2\pi A m}{\Delta}\right) e^{j16p\phi} \end{aligned}$$

The sum over  $p$  above is evaluated in Appendix C. Inserting the result of Appendix C, interchanging the order of summations, and using Euler's rule yields the second part of Eq. (13).

Now return to the second part of Eq. (B-1):

$$\begin{aligned} W &= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{\substack{r=0 \\ r \neq n}}^{N-1} E\{x[n]x[r]\} \exp\left(\frac{-j2\pi k(n-r)}{N}\right) \\ &= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{\substack{r=0 \\ r \neq n}}^{N-1} R_e(nT, rT - nT) \exp\left(\frac{-j2\pi k(n-r)}{N}\right) \end{aligned}$$

where  $R_e(\cdot)$  is defined in Eq. (A-2). Evaluating this further,

$$\begin{aligned} W &= \\ &- \frac{\Delta^2}{4\pi^2} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{(-1)^{l+m}}{lm} \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} (l^2 + m^2)\right) \\ &\times \left[ \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} \exp\left(\frac{j2\pi A}{\Delta} [l \sin(\omega_0 T n + \phi) \right. \right. \\ &\quad \left. \left. + m \sin(\omega_0 T r + \phi)]\right) \exp\left(\frac{j2\pi k(n-r)}{N}\right) \right. \\ &\quad \left. - \frac{1}{N^2} \sum_{n=0}^{N-1} \exp\left(j \frac{2\pi A(l+m)}{\Delta} \sin(\omega_0 T n + \phi)\right) \right] \end{aligned}$$

Note that the second term inside the big brackets is independent of the bin number  $k$ . Further separating the terms inside the big brackets, one can write

$$W = S_Q + N_{Q_3} \quad (\text{B-2})$$

Consider the  $N_{Q_3}$  term (this is the term independent of  $k$ ). Employing the Jacobi-Anger formula and evaluating the sum over  $n$  as before,

$$N_{Q_3} =$$

$$\begin{aligned} & \frac{\Delta^2}{4\pi^2 N} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{(-1)^{(l+m)} \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} (l^2 + m^2)\right)}{lm} \\ & \times \sum_{k=-\infty}^{\infty} J_{16k} \left( \frac{2\pi A}{\Delta} (l+m) \right) e^{j16\phi k} \end{aligned}$$

The results of Appendix C can be used to evaluate the infinite sum over  $k$ . Again, after manipulating the definition of  $F(\cdot)$  in Appendix C as detailed above, one finally obtains the third part of Eq. (13).

Finally, consider  $S_Q$  from Eq. (B-2):

$$S_Q = -\frac{\Delta^2}{4\pi^2} \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{(-1)^{l+m}}{lm}$$

$$\begin{aligned} & \times \exp\left(-\frac{2\pi^2 \sigma^2}{\Delta^2} (l^2 + m^2)\right) \\ & \times \left[ \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} \exp\left(\frac{j2\pi A}{\Delta} [l \sin(\omega_0 T n + \phi) \right. \right. \\ & \left. \left. + m \sin(\omega_0 T r + \phi)]\right) \exp\left(\frac{-j2\pi k(n-r)}{N}\right) \right] \end{aligned}$$

Call the sum in the big brackets  $V$ , and employ the Jacobi-Anger formula. Using the values of  $\omega_0 T$  and  $k$  from Section III, and assuming that  $N$  is a power of 2 ( $N \geq 16$ ), the value of  $V$  reduces to

$$\begin{aligned} V &= \left( \sum_{p=-\infty}^{\infty} J_{3+16p} \left( \frac{2\pi A l}{\Delta} \right) e^{j16\phi p} \right) \\ & \times \left( \sum_{q=-\infty}^{\infty} J_{-3+16q} \left( \frac{2\pi A m}{\Delta} \right) e^{j16\phi q} \right) \end{aligned}$$

Since  $J_n(x) = (-1)^n J_n(-x)$  and  $\pm 3 + 16p$  is odd, one can change the double-sided infinite sums to single-sided infinite sums. By changing the index on  $q$  to  $-q$ , recalling that  $J_{-n}(x) = (-1)^n J_n(x)$ , and using the formula in Appendix C, one gets Eq. (14).

## Appendix C

### Evaluating an Infinite Bessel Sum

Define

$$F(x, y, z, t) = \sum_{k=-\infty}^{\infty} e^{jxk} J_{y+zk}(t)$$

where  $x$  and  $y$  are integers and  $z \neq 0$ . Using the integral definition of the Bessel function of integer order [7],

$$F(x, y, z, t) =$$

$$\sum_{k=-\infty}^{\infty} e^{jxk} \frac{j^{(-y+zk)}}{\pi} \int_0^{\pi} \cos(y\theta + zk\theta) e^{jt \cos(\theta)} d\theta$$

After changing the order of integration and summation, one can apply Euler's rule to express  $\cos(\cdot)$  in terms of complex exponentials. Next, equate the infinite sums of complex exponentials to infinite sums of Dirac delta functions as in Eq. (A-3). Only a finite number of delta functions will remain inside the limits of integration. Note that

the remainder of this analysis assumes (for simplicity) that no delta functions lie on the limits of integration. For the DFT example in this article, this condition is satisfied. Using the sifting and scaling properties of delta functions, one finally obtains

$$F(x, y, z, t) =$$

$$\frac{j^{-y}}{z} \sum_{m=0}^{\lfloor z/2 \rfloor - 1} \left[ \exp(j(t\Theta_m + y\theta_m)) + \exp(j(t\Xi_m - y\xi_m)) \right]$$

where

$$\Theta_m = \frac{\pi}{2} - \frac{x}{z} + \frac{2\pi}{z} \left( \left\lfloor \frac{x}{2\pi} - \frac{z}{4} \right\rfloor + m \right)$$

$$\xi_m = \frac{\pi}{2} + \frac{x}{z} - \frac{2\pi}{z} \left( \left\lfloor \frac{x}{2\pi} - \frac{z}{4} \right\rfloor - m \right)$$

and  $\Theta_m = \cos(\theta_m)$  and  $\Xi_m = \cos(\xi_m)$ .

## Appendix D

### Truncating an Infinite Sum

This appendix considers bounds on the error introduced by truncating the following infinite sum:

$$\begin{aligned}
 x &= \sum_{l=1}^{\infty} \frac{e^{-\alpha^2 l^2}}{l^n} F(l) \\
 &= \sum_{l=1}^L \frac{e^{-\alpha^2 l^2}}{l^n} F(l) + \sum_{l=L+1}^{\infty} \frac{e^{-\alpha^2 l^2}}{l^n} F(l) \\
 &= \hat{x} + \text{error}
 \end{aligned}$$

where  $n \geq 1$  and  $|F(l)| \leq 1$ . The error can be bounded as follows:

$$|\text{error}| \leq \sum_{l=L+1}^{\infty} \frac{e^{-\alpha^2 l^2}}{l^n} |F(l)| \leq \sum_{l=L+1}^{\infty} \exp(-\alpha^2 l^2)$$

Let  $\alpha^2 = 1/2\sigma^2$ . Then,

$$|\text{error}| \leq \sqrt{2\pi\sigma^2} \sum_{l=L+1}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{l^2}{2\sigma^2}}$$

The above sum can be visualized as the integral from  $x = L$  to  $x = \infty$  of a discontinuous step-like function whose value is constant over the interval between two adjacent integers. The value over an interval is equal to the Gaussian density evaluated at the right-most portion of the interval. A little thought will show that this integral is strictly less than the integral of a Gaussian distribution from  $x = L$  to  $x = \infty$ . Therefore,

$$|\text{error}| < \frac{\sqrt{\pi}}{2\alpha} \text{erfc}(\alpha L)$$

where  $\text{erfc}(\cdot)$  is the complementary function defined in [7].

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